PHYSICSAccess Short Report
ISSN Online: 2756-3898, ISSN Print: 2714-500X https://doi.org/10.47514/phyaccess.2025.5.1.008

# General Relativistic Study of Motion of Comets Particles Elliptical Gravitational Field

**Timothy Bulus[1], Chifu E Ndikilar[2], Halliru Ibrahim[1] and Dayyabu Tafida[1]**

[1] Department of Science Education, Physics Education, Kaduna State University, Kaduna State, Nigeria
[2] Department of Physics, Federal University Dutse, Jigawa State, Nigeria

Corresponding E-mail: timbulusasmau@gmail.com

Received 11-02-2025
Accepted for publication 27-03-2025
Published 28-03-2025

## Abstract

Comets are small bodies consisting of aggregates of ice mixed with rock and dust. They are usually influenced by galactic forces and stellar encounters, and may have contributed to the formation of ice giants and transport gases across the solar system. Their orbits vary widely, and their frozen water sublimates around 3 AU, forming a coma, though some remain active beyond this distance. In this study, the line element in the gravitational field due to a static and ellipsoidal isolated gravitating mass point was used to study the motion of comets, and the relativistic equation of motion of an ellipsoidal mass was obtained via the metric tensor, affine connections and geodesics equation. The results show that the explicit equations of comets along the equatorial plane are second-order differential equations similar to reported results in the literature for different gravitational fields.

Keywords: General Relativity; Comets; Elliptical Orbit; Metric Tensor.

## I. Introduction

The motion of comets from the sun is primarily affected by the vertical component of the galactic field, as well as encounters with stars and giant molecular clouds [1]. The ice giants (Uranus and Neptune) are hypothesized to have been formed from comets that served as fundamental building blocks, similar to the formation of Jupiter and Saturn, the gas giants. These comets demonstrate the potential to carry gaseous materials from the outer solar system to the inner planets [2]. Several efforts have been made to address the complexities of comet dust, which involves the size, composition, shape, and structure of its particles using data obtained from its transport mechanisms [3]. To broadly understand the description of a regular comet, the elliptical symmetry in its general form is taken and applied to a particle-filled comet using a mathematical model.

Comets represent a prevalent constituent of the solar system, wherein their orbital trajectories may, in certain instances, exhibit perihelion distances that are less than the solar radius, while in numerous other situations, they may possess aphelion distances sufficiently extensive to render them unbound from the gravitational influence of the sun [1]. Also, as they orbit close to the inner Solar System, a coma forms when its frozen water ($H_2O$) starts to sublimate at approximately three astronomical units (AU) from the Sun. However, some comets exhibit measurable activity beyond this sublimation point, with nearly one-third of observed comets displaying activity past the 3 AU mark. While water ice typically sublimates around this distance, the process can still contribute to some activity beyond it [4].

The General Theory of Relativity is regarded as the most accepted Modern Theory of Gravitation introduced by Einstein in 1916. In this article, the motion of comet particles within elliptical gravitational fields is studied in the framework of the General Theory of Relativity.

## II. Metric Tensor of an Elliptical Gravitational Mass Point

Elliptical fields of stars or objects can be considered elliptically symmetric such that the structure, matter and field of such an object can be defined in terms of elliptically symmetric space metric.

The metric of a gravitating mass point in its complete form in elliptical polar coordinates is given by (1) [5].

$$ds^2 = \left(1-\frac{2m}{r}\right)c^2dt^2 - \left(1-\frac{2m}{r}\right)^{-1}\left(\frac{r^2+a^2cos^2\theta}{r^2+a^2}\right)dr^2 - (r^2+a^2cos^2\theta)d\theta^2 - (r^2+a^2)sin^2\theta d\phi^2 \qquad (1)$$

Where 'a' is a constant in the xy-surface. Here, one can say that the meaning of m is as defined in Schwarzschild's equation. Putting $m=\frac{GM}{c^2}$, we can see that for a large value of r, $g \approx \frac{GM}{r^2}$ for M to be the mass of the central body. Then $c = G = 1$ from relativistic units can warrant us to have $m = M$, which is dimensionally correct [6]. When $r < a$, the metric is the internal metric of the elliptical field with an incompressible liquid. At the surface $r = a$, the metric coincides with a mass point and the outer metric.

From (1), we can deduce the contra-variant metric tensor using the quotient theorem of tensor analysis and formulate the non-vanishing coefficients of affine connection in their complete form as:

$$\Gamma^0_{01} = \frac{\rho'}{2} \qquad (2)$$

$$\Gamma^0_{02} = \frac{\dot\rho}{2} \qquad (3)$$

$$\Gamma^1_{00} = \frac{\rho' e^\rho}{2e^\sigma}\left(\frac{r^2+a^2}{r^2+a^2cos^2\theta}\right) \qquad (4)$$

$$\Gamma^1_{11} = \frac{\sigma'}{2} + \frac{r}{r^2+a^2cos^2\theta} - \frac{r}{r^2+a^2} \qquad (5)$$

$$\Gamma^1_{12} = \frac{\dot\sigma}{2} - \frac{a^2\sin 2\theta}{2} \qquad (6)$$

$$\Gamma^1_{22} = \left(\frac{-r(r^2+a^2)}{e^\sigma(r^2+a^2cos^2\theta)}\right) \qquad (7)$$

$$\Gamma^1_{33} = \frac{-r(r^2+a^2)sin^2\theta}{e^\sigma(r^2+a^2cos^2\theta)} \qquad (8)$$

$$\Gamma^2_{00} = \frac{\dot\rho e^\rho}{2(r^2+a^2cos^2\theta)} \qquad (9)$$

$$\Gamma^2_{11} = \frac{-\dot\sigma e^\sigma}{2} + \frac{e^\sigma a^2\sin 2\theta}{2(r^2+a^2cos^2\theta)(r^2+a^2)} \qquad (10)$$

$$\Gamma^2_{12} = \frac{r}{(r^2+a^2cos^2\theta)} \qquad (11)$$

$$\Gamma^2_{22} = \frac{-a^2\sin 2\theta}{(r^2+a^2cos^2\theta)} \qquad (12)$$

$$\Gamma^2_{33} = \frac{-(r^2+a^2)\sin 2\theta}{2(r^2+a^2cos^2\theta)} \qquad (13)$$

$$\Gamma^3_{13} = \frac{r}{r^2+a^2} \qquad (14)$$

$$\Gamma^3_{23} = \cot\theta \qquad (15)$$

## III. Motion of Comet Particles in the Vicinity of an Ellipsoidal Mass

The general relativistic equation of motion for particles of non-zero rest mass in a gravitational field is given in [7].

$$\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\alpha\lambda}\frac{dx^\alpha}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \qquad (16)$$

Where $\tau$ is the proper time.

Therefore, the equations of motion are given explicitly as follows.

When $\sigma = 0, x^0 = ct, x^1 = r, x^2 = \theta, x^3 = \phi$

$$\frac{d^2x^0}{d\tau^2} + \Gamma^0_{\alpha\lambda}\frac{dx^\alpha}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \qquad (17)$$

$$\frac{d^2(ct)}{d\tau^2} + \Gamma^0_{01}\frac{dx^0}{d\tau}\frac{dx^1}{d\tau} + \Gamma^0_{02}\frac{dx^0}{d\tau}\frac{dx^2}{d\tau} = 0 \qquad (18)$$

$$\frac{d^2(ct)}{d\tau^2} + \Gamma^0_{01}\frac{d(ct)}{d\tau}\frac{dr}{d\tau} + \Gamma^0_{02}\frac{d(ct)}{d\tau}\frac{d\theta}{d\tau} = 0 \qquad (19)$$

$$c\ddot t + \Gamma^0_{01}c\dot t\dot r + \Gamma^0_{02}c\dot t\dot\theta = 0 \qquad (20)$$

Substituting (2) and (3) into (20) gives (21).

$$c\ddot t + v'c\dot t\dot r + \dot v c\dot t\dot\theta = 0 \qquad (21)$$

Equation (20) is the time equation of motion for comets in this gravitational field.

Similarly, setting $\sigma = 1, x^0 = ct, x^1 = r, x^2 = \theta, x^3 = \phi$

$$\frac{d^2x^1}{d\tau^2} + \Gamma^1_{\alpha\lambda}\frac{dx^\alpha}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \qquad (22)$$

$$\frac{d^2r}{d\tau^2} + \Gamma^1_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \Gamma^1_{11}\frac{dx^1}{d\tau}\frac{dx^1}{d\tau} + 2\Gamma^1_{12}\frac{dx^1}{d\tau}\frac{dx^2}{d\tau} + \Gamma^1_{22}\frac{dx^2}{d\tau}\frac{dx^2}{d\tau} + \Gamma^1_{33}\frac{dx^3}{d\tau}\frac{dx^3}{d\tau} = 0$$

$$\ddot r + \Gamma^1_{00}\dot t^2 + \Gamma^1_{11}\dot r^2 + 2\Gamma^1_{12}\dot r\dot\theta + \Gamma^1_{22}\dot\theta^2 + \Gamma^1_{33}\dot\phi^2 = 0 \qquad (23)$$

Substituting (4), (5), (6), (7) and (8) into (23), we get (24).

$$\ddot r + \frac{\rho' e^\rho}{2e^\sigma}\left(\frac{r^2+a^2}{r^2+a^2cos^2\theta}\right)\dot t^2 + \left(\frac{\sigma'}{2} + \frac{r}{r^2+a^2cos^2\theta} - \frac{r}{r^2+a^2}\right)\dot r^2 + 2\left(\frac{\dot\sigma}{2} - \frac{a^2\sin 2\theta}{2}\right)\dot r\dot\theta - \frac{r(r^2+a^2)}{e^\sigma(r^2+a^2cos^2\theta)}\dot\theta^2 - \frac{r(r^2+a^2)sin^2\theta}{e^\sigma(r^2+a^2cos^2\theta)}\dot\phi^2 = 0 \qquad (24)$$

This is the radial equation of motion of comet particles in this gravitational field.

Putting $\sigma = 2$, and $\sigma = 3$ in (16), the polar and azimuthal equations of motion in the inner gravitational field of an ellipsoidal star are obtained respectively as (25) and (26).

$$\ddot\theta + \frac{\dot\rho e^\rho}{2(r^2+a^2cos^2\theta)}\dot t^2 + \left(\frac{-\dot\sigma e^\sigma}{2} + \frac{e^\sigma a^2\sin 2\theta}{2(r^2+a^2cos^2\theta)(r^2+a^2)}\right)\dot r^2 + 2\frac{r}{(r^2+a^2cos^2\theta)}\dot r\dot\theta - \frac{a^2\sin 2\theta}{(r^2+a^2cos^2\theta)}\dot\theta^2 - \frac{(r^2+a^2)\sin 2\theta}{2(r^2+a^2cos^2\theta)}\dot\phi^2 = 0 \qquad (25)$$

and

$$\ddot\phi + 2\frac{r}{r^2+a^2}\dot r\dot\phi + 2\cot\theta\,\dot\theta\dot\phi = 0 \qquad (26)$$

Equation (24) is space-time related, which is similar to the Schwarzschild radial equation of motion with only the inclusion of adjustable parameters such as $(a\cos\theta)^2$.

Note that when $\sigma = 0$, the geodesic equation of motion is explicitly radial, polar and time coordinate derivative of proper time, which is different from the Schwarzschild. Geodesic equation $\sigma = 3$ is purely space-related, while the rest are space-time related.

Now, considering that comet particles are moving in the equatorial plane $(\theta = {}^{\pi}/_{2})$ of the ellipsoidal star, (1) reduces to (27).

$$c^2 = \left(1-\frac{2m}{r}\right)c^2\dot t^2 - \left(1-\frac{2m}{r}\right)^{-1}\left(\frac{r^2}{r^2+a^2}\right)\dot r^2 - -(r^2+a^2)\dot\phi^2 \qquad (27)$$

Explicit expressions for $\dot t$ and $\dot\phi$ are obtained by solving the geodesic equations (21) and (26).

From the time like geodesic equation, at the equatorial, (21) becomes.

$$\frac{\ddot{t}}{\dot{t}} + v'\dot{r} = 0 \quad (28)$$

Taking the integral of (28) with respect to t gives (29).

$$\dot{t} = e^{A-v} \quad (29)$$

Where A is an arbitrary constant.

Letting $e^A = K$ and recall that $e^v = \left(1 - \frac{2m}{r}\right)$, (29) becomes

$$\dot{t} = K\left(1 - \frac{2m}{r}\right)^{-1} \quad (30)$$

Considering the geodesic equation (26) at the equatorial plane, it can be shown that,

$$\dot{\phi} = \frac{B}{(r^2+a^2)} \quad (31)$$

Where B is an arbitrary constant.

Writing (27) in terms of (30) and (31) gives (32).

$$c^2 = \left(1 - \frac{2m}{r}\right)c^2\left(K\left(1 - \frac{2m}{r}\right)^{-1}\right)^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{r^2}{r^2+a^2}\right)\dot{r}^2 - -(r^2+a^2)\left(\frac{B}{(r^2+a^2)}\right)^2 \quad (32)$$

Equation (32) can be written in a simplified form as (33).

$$\dot{r}^2 = \frac{2mc^2(r^2+a^2)}{r^3} - \frac{B^2}{r^2} + \frac{2mB^2}{r^3} \quad (33)$$

Differentiating (33) with respect to r gives (34).

$$\ddot{r} = -mc^2\left(\frac{1}{r^2} + \frac{3a^2}{r^4}\right) + \frac{B^2}{r^3} - \frac{3mB^2}{r^4} \quad (34)$$

Expressing $\ddot{r}$ on the left-hand side of (34) as a function of $\phi$ using the substitution $r = \frac{1}{u(\phi)}$, yields (35).

$$\ddot{r} = \frac{-B^2u^2}{(1+u^2a^2)^2}\frac{d^2u}{d\phi^2} \quad (35)$$

Thus, (34) can be written as (36).

$$\frac{d^2u}{d\phi^2} + u(1+u^2a^2)^2 = \left[\frac{mc^2}{B^2} + 3mu^2\left(1 + \frac{a^2c^2}{B^2}\right)\right](1 + u^2a^2)^2 \quad (36)$$

Equation (36) is the equation of motion of comets along the equatorial plane of an ellipsoidal star. The term ‘a’ distinguished this equation from the Schwarzschild spherical theory for a spherical mass, which shows that an ellipsoidal gives the best gravitational equation for non-spherical objects such as comets.

## IV. Conclusion

The space metric due to a static and ellipsoidal isolated gravitating mass point is being exploited to deduce the equation of motion of comets in the vicinity of an ellipsoidal mass. These equations can be solved to give a profound description of a Comet. This study can reveal the true mechanical nature of constituent particles in comets that are elliptical when extended.

## References


[1] A. Carussi and G. B. Valsecchi. “Dynamics of comets Chaos, Resonance and Collective Dynamical Phenomena in the Star”. Netherlands; S. FErraz-mello (ed.), pp. 255-268, 1992.

[2] M. Wesolowski, P. Gronkowski, and I. Tralle. ”Selected mechanisms of matter ejection out of the cometary nuclei”. Icarus, vol. 338, pp. 113546, 2020.

[3] A. H. Delsemme. “Kuiper Prize lecture: Cometary origin of the biosphere”. Icarus vol. 146, pp. 313-325, 2000.

[4] M. Womack, G. Sarid, and K. Wierzchos. “CO and Other Volatiles in Distantly Active Comets”. Pub. of the Astro. Soc. of the Pacific, vol. 129, 031001, pp. 1-20, 2017.

[5] B. Nikouravan. “Schwarzschild-like solution for ellipsoidal celestial objects”. IJPS, vol. 6, no. 6, pp. 1426-1430, 2011.

[6] R. Wolfgang. “Relativity special, General, and Cosmological”. Oxford University Press, 2006.

[7] E. N. Chifu and L.W Lumbi. "General Relativistic Equations of Motion for Test Particles Exterior to Astrophysically Real or Hypothetical Spherical Distribution of mass Whose Tensor Field varies with Azimuthal Angle only”. Cont. J. of App. Sci., vol. 3, pp. 32-38, 2008a.